Remarks on the Mixed Multipole Internal Conversion Electron Correlation Parameters*

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The internal conversion electron correlation parameters are calculated for the general K-shell (mixed) multipole conversion. Phase conventions are critically examined, and the sign of the mixed electric-magnetic conversion proposed by Church, Weneser, and Schwarzschild is verified. Phase conventions in the literature for the Dirac-Coulomb continuum wave functions are shown to be ambiguous, and a suitable standard convention is proposed.

I. INTRODUCTION AND SUMMARY

THE usefulness of internal conversion angular correlations as a technique for nuclear spectroscopy was pointed out in 1949 by Gardner, and by Fierz, and further developed by many others. This work was surveyed, and extended to $E(L+1)-ML$ mixed multipoles, in the review paper of Biedenharn and Rose.[1] The experiments envisaged in these calculations were difficult, and the possibility of observing the more complicated correlations seemed experimentally remote. Subsequent theoretical efforts were in consequence concentrated largely on the importance of electron penetration effects for internal conversion.[2–4]

This situation has changed markedly in the last few years with the development of high resolution conversion spectrometers and solid-state counter techniques. This increased experimental interest has led to a corresponding interest in the details of the theoretical calculations, including the mixed conversion correlations whose treatment was but briefly given in Ref. 1. The situation became rather more acute with the investigation of Church, Schwarzschild, and Weneser[5] into the sign of the mixed (electric-magnetic) conversion correlation parameters.

The calculation of the mixed multipole correlation parameters is complicated at best, and for sign determinations the tensor parameter method (because of the use of implicit phase conventions) is not the optimal technique. In the present paper an *ab initio* calculation (an extension of the Green function method used earlier for pure multipoles[6]) is employed to calculate in detail the general (K shell) mixed multipole results [Eqs. (25), (31), and (36)]. Results of this generality do not seem to have been previously given in the literature.

The results obtained are specialized to $E(L+1)-ML$ mixtures and critically compared to previous calculations. The sign found by Church, Schwarzschild, and Weneser[5] is verified, and the corresponding result of Biedenharn and Rose[1] is found to be in error.

As discussed in the concluding section, a major source of confusion in the internal conversion calculations stems from an ambiguity in the definition of the continuum relativistic Coulomb functions in the literature. Suitable standard conventions are presented in the detailed summary of the Dirac-Coulomb functions given in Sec. IV.

II. CONVERSION ELECTRON PARTICLE PARAMETERS

We shall present here a brief but complete *ab initio* calculation of the particle parameters in internal conversion. In order that the results be as concise as possible we shall consider explicitly the (converted) gamma-ray emission, and define particle parameters *directly* with respect to this gamma-ray angular distribution. The necessity for such a procedure results from the fact that any definition of the particle parameters in terms of some shorter formalism—for example, by using the technique of tensor parameters (e.g., BR p. 735ff)—is not an *ab initio* calculation since such a formalism introduces various phase conventions *implicitly*, and this is to be avoided in a critical discussion. To be helpful, and explicit, references to equivalent steps in earlier calculations will be cited.

In order that the phase assumptions be minimal we shall present a technique whereby *all* phases are defined relative to the plane-wave result. The sole phase convention to be assumed is that of Condon-Shortley for

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[1] L. C. Biedenharn and M. E. Rose, Rev. Mod. Phys. **25**, 729 (1953) (referred to as BR henceforth). References to the early literature may be found in this paper.

[2] G. Kramer, Z. Physik **146**, 187 (1956); **147**, 628 (1957).

[3] M. E. Rose and T. A. Green, Phys. Rev. **110**, 105 (1958).

[4] E. L. Church and J. Weneser, Phys. Rev. **104**, 1382 (1956).

[5] E. L. Church, A. Schwarzschild, and J. Weneser, Phys. Rev. **133**, B35 (1964). Reference to the literature on recent experimental investigations may be found in this paper.

[6] M. E. Rose, L. C. Biedenharn, and G. B. Arfken, Phys. Rev. **85**, 5 (1952). Hereinafter referred to as RBA.

the $Y_l^m(\vartheta\varphi)$ and for the Wigner coefficients. The method of Green's functions will be used, and all Green's functions will be based on the (outgoing) free wave Green's function,

$$G_\infty = e^{ikR}/4\pi R, \quad R \equiv |\mathbf{r}_1 - \mathbf{r}_2|, \tag{1}$$

which has the familiar expansion,

$$G_\infty = ik \sum_{lm} h_l(kr_>) j_l(kr_<) Y_l^m(\hat{r}_1) Y_l^{m*}(\hat{r}_2). \tag{2}$$

Taking the limit of $|r_2| \to \infty$, one derives from this the Bauer formula, $(\hat{k} = -\hat{r}_2)$

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{lm} [4\pi(2l+1)]^{1/2} (i^l Y_l^m(\hat{r})) D_{m,0}{}^l(\hat{k}) j_l(kr). \tag{3}$$

One further convention ("Convention T") is used. This is the convention introduced in BR p. 736 that all angular basis functions will be phased such that under time-reversal one finds: $T|jm\rangle = (-)^{j-m}|j-m\rangle$. (Here T is the time reversal operator $T \equiv -i\sigma_y K_0$ for spin $\frac{1}{2}$, $T = K_0$ for integer spin where $K_0 \equiv$ complex conjugation.) {The choice $(-)^{j-m}$ [rather than $(-)^{j+m}$ which is equally allowed] necessitates the choice that $-i\boldsymbol{\sigma}_y$ be used, in order that the basis functions $|\frac{1}{2}, \pm\frac{1}{2}\rangle$ be chosen as: $|\frac{1}{2}, \frac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\frac{1}{2}, -\frac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.} This convention has the advantage that almost all phases drop out of the formulas.

Gamma-Ray Emission

The emission of gamma rays in the nuclear transition $J_iM_i \to J_fM_f$ is described by the vector potential (we use the radiation gauge $A_4 = 0$) obtained from interaction of the (dyadic) Green's functions with the nuclear currents $(\equiv \mathbf{j}_N)$.

The required Green's function is immediate:

$$\mathbf{G}_\infty = \mathbf{1}G_\infty = G_\infty \sum_m |1,m\rangle\langle 1,m|, \tag{4}$$

where $|1,m\rangle$ are the basis vectors for spin 1 ("spherical vector basis"), phased in convention T. Using Eq. (2) and the definition [BR Eqs. (53) and (105)]:

$$\mathfrak{Y}_{JL}{}^M \equiv \sum_m C^L{}_{M-m}{}^1{}_m{}^J (i^L Y_L{}^{M-m}(\vartheta\varphi)) |1,m\rangle, \tag{5}$$

one finds that:

$$\mathbf{G}_\infty = ik \sum_{JLM} h_L(kr_<) j_L(kr_>) \mathfrak{Y}_{JL}{}^M(\hat{r}_1) (\mathfrak{Y}_{JL}{}^M(\hat{r}_2))^\dagger. \tag{6}$$

We rearrange this latter sum (in a manner formally equivalent to a rotation) to define the magnetic (m), electric (e), and longitudinal (l) vector multipole functions:

$$\mathbf{A}_{Lm}{}^M = f_L(kr) \mathfrak{Y}_{LL}{}^M, \tag{7a}$$

$$\mathbf{A}_{Le}{}^M = [(L+1)/(2L+1)]^{1/2} f_{L-1}(kr) \mathfrak{Y}_{L,L-1}{}^M - [L/(2L+1)]^{1/2} f_{L+1}(kr) \mathfrak{Y}_{L,L+1}, \tag{7b}$$

$$\mathbf{A}_{Ll}{}^M = [L/(2L+1)]^{1/2} f_{L-1}(kr) \mathfrak{Y}_{L,L-1}{}^M + [(L+1)/(2L+1)]^{1/2} f_{L+1}(kr) \mathfrak{Y}_{L,L+1}{}^M, \tag{7c}$$

where the $f_L(kr)$ are spherical Bessel functions.

Finally then:

$$\mathbf{G}_\infty = ik \sum_{LM\pi=m,e,l} \mathbf{A}_{L\pi}{}^M(\text{out},>) (\mathbf{A}_{L\pi}{}^M(\text{st},<))^\dagger. \tag{8}$$

(Note that the Hermitian conjugate *must* now be applied to the standing wave solution, since the outgoing radial function is not real.)

The gamma emission is then described, outside sources, by the vector potential:

$$\mathbf{A}_{\text{emitted}} = ik \sum_{\substack{LM\\ \pi=e,m}} \mathbf{A}_{L\pi}{}^M(\text{out}) \times \langle J_fM_f | \mathbf{j}_N \cdot \mathbf{A}_{L\pi}{}^{M\dagger}(\text{st}) | J_iM_i \rangle. \tag{9}$$

The angular distribution is obtained from:

(a) $$\mathbf{A}_{L\pi=e,m}{}^M(\text{out}) \sim \left(\frac{e^{ikr}}{ikr}\right) \sum_{P=\pm1} [(2L+1)/8\pi]^{1/2} \times (-P)^{\sigma(\pi)} D_{P,M}{}^L(\mathcal{R}^{-1}) |1,P\rangle, \tag{10}$$

(b) $$\mathbf{A}_{\text{emitted}} \sim \sum_{\substack{LM,P\\ \pi=e,m}} \left(\frac{e^{ikr}}{ikr} |1,P\rangle\right) \times [(2L+1)/8\pi]^{1/2} D_{P,M}{}^L(\mathcal{R}^{-1}) (-P)^{\sigma(\pi)} \times \langle J_fM_f | \mathbf{j}_N \cdot \mathbf{A}_{L\pi}{}^{M\dagger}(\text{st}) | J_iM_i \rangle, \tag{11}$$

where

$$\sigma(\pi) \equiv 1, \quad \pi = m$$
$$\equiv 0, \quad \pi = e$$

as in BR p. 751.

The observation of a gamma ray, along the direction $\mathbf{k}$ (Euler angles $\mathcal{R}$ with respect to the reference axes) is thus given by the probability:

$$W_\gamma(\vartheta) \propto \sum_{\substack{P\\ M_f,M_i}} \left| \sum_{L\pi M} (2L+1)^{1/2} D_{P,M}{}^L(\mathcal{R}^{-1}) (-P)^{\sigma(\pi)} \times \langle J_fM_f | \mathbf{j}_N \cdot \mathbf{A}_{L\pi}{}^{M\dagger} | J_iM_i \rangle \right|^2, \tag{12}$$

where we have summed over both $P = \pm 1$ and over the initial and final nuclear magnetic quantum numbers (M_iM_f) to obtain an unpolarized measurement.

Conversion Electron Emission

The paradigm above shows that one requires both Green's function[6] and the plane-wave result (which are not independent) for the electron waves in the presence of the nuclear Coulomb field. [We adopt the convenient artifice that the 'plane' wave for a Coulomb field exists, by including lnpr terms (where required) in the definition of pr.] In order to make the essentials of the angular correlation argument as clear as possible we shall, however, relegate the necessary discussion of the electron wave functions to a subsequent section (Sec. IV). As will be shown there, the Coulomb 'plane' wave for an electron having helicity $\tau(=\pm\frac{1}{2})$ and asymptotic direction $\hat{p}$ is given by:

$$|\mathbf{p},\tau\rangle = 2\pi[p(E+1)]^{-1/2} \times\sum_{\kappa}|\kappa|^{1/2}e^{i\Delta_\kappa}[-S(\kappa)]^{\tau-1/2}D_{M,\tau}{}^{j}(\hat{p})|\mathrm{st};\kappa\mu\rangle, \quad (13)$$

where the 'standing wave' central field solutions are

$$|\mathrm{st};\kappa\mu\rangle \equiv \begin{pmatrix} S(\kappa)f_\kappa(pr)\Phi_{-\kappa}{}^{\mu} \\ g_\kappa(pr)\Phi_\kappa{}^{\mu} \end{pmatrix} = i^{l(\kappa)}\begin{pmatrix} -if_\kappa(pr)\chi_{-\kappa}{}^{\mu} \\ g_\kappa(pr)\chi_\kappa{}^{\mu} \end{pmatrix}, \quad (14)$$

using both the Pauli central field spinors $\chi_\kappa{}^\mu$ and the central field spinors $\Phi_\kappa{}^\mu$ of convention T. The f_κ, g_κ are the radial functions of Refs. 3 and 4. {Equation (13) is just BR Eq. (84), *except* that the phase $i^{-l(\kappa)}$ in BR Eqs. (84) [as well as in Eqs. (82), (85), (87), (88)] should be replaced by $i^{+l(\kappa)}$. For ν even Eqs. (85), (87), (88) are, however, nonetheless correct.} Using *outgoing* radial solutions for f, g one defines the solutions $|\mathrm{out};k\mu\rangle$.

The desired Green's function [RBA Eqs. (17), (18)] is

$$\mathcal{G} = (-\pi i)\sum_{\kappa\mu}|\mathrm{out},>;\kappa\mu\rangle\langle\mathrm{st},<;\kappa\mu|. \quad (15)$$

The wave function for the emitted electron is then found to be

$$\psi_{\mathrm{emitted}} \equiv \int \mathcal{G}\mathcal{H}_{\mathrm{int}}|\kappa_i\mu_i\rangle dv, \quad (16)$$

with $\mathcal{H}_{\mathrm{int}}$ being the interaction, i.e.,

$$\mathcal{H}_{\mathrm{int}} \equiv \mathbf{j}_{\mathrm{el}}\cdot\mathbf{G}_\infty\cdot\mathbf{j}_N = \sum_{LM\pi=e,m}\mathbf{j}_{\mathrm{el}}\cdot\mathbf{A}_{L\pi}{}^{M}(\mathrm{out},>)\mathbf{j}_N\cdot\mathbf{A}_{L\pi}{}^{M\dagger}(\mathrm{st},<). \quad (17)$$

It follows that:

$$\psi_{\mathrm{emitted}} \propto \sum_{\substack{\kappa\mu\\ LM\pi}}|\mathrm{out};\kappa\mu\rangle\langle\kappa\mu|\mathbf{j}_{\mathrm{el}}\cdot\mathbf{A}_{L\pi}{}^{M}(\mathrm{out})|\kappa_i\mu_i\rangle \times\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L\pi}{}^{M\dagger}(\mathrm{st})|J_iM_i\rangle. \quad (18)$$

(This assumes no electron penetration, i.e., a point nucleus.) Using next the asymptotic form of the solution $|\mathrm{out};\kappa\mu\rangle$ one finds:

$$|\mathrm{out};\kappa\mu\rangle \sim \left(\frac{e^{ipr}}{ipr}\right)\sum_{\tau=\pm\frac{1}{2}}[|\kappa|/4\pi]^{1/2} \times e^{i\Delta_\kappa}[-S(\kappa)]^{\tau-\frac{1}{2}}D_{\tau,\mu}{}^{j}(R^{-1})D_\tau, \quad (19)$$

where

$$D_\tau = \begin{bmatrix} -p/(E+1) \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \tau=\tfrac{1}{2}$$

$$= \begin{bmatrix} 0 \\ p/(E+1) \\ 0 \\ 1 \end{bmatrix} \quad \tau=-\tfrac{1}{2}.$$

The observation of an electron with helicity τ moving outward in the direction $\hat{p}$ (Euler angles R with respect to axes) has the probability:

$$W_{\mathrm{c.e.}}(\theta) \propto \sum_{\substack{\tau\mu_i\\ M_iM_f}}\Big|\sum_{\substack{\kappa\mu\\ LM\pi=e,m}}([-S(\kappa)]^{\tau-\frac{1}{2}}D_{\tau,\mu}{}^{j}(R^{-1})e^{i\Delta_\kappa}|\kappa|^{1/2}) \times\langle\kappa\mu|\mathbf{j}_{\mathrm{el}}\cdot\mathbf{A}_{L\pi}{}^{M}(\mathrm{out})|\kappa_i\mu_i\rangle \times\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L\pi}{}^{M\dagger}(\mathrm{st})|J_iM_i\rangle\Big|^2. \quad (20)$$

[The observation of an unpolarized electron corresponds to the sum over τ given above; in addition the K-shell magnetic quantum number (μ_i) and the nuclear magnetic quantum numbers (M_iM_f) are averaged.]

The Particle Parameters

Since the emission process has precisely the same form in the two distributions given above a direct comparison is possible between the two processes and the particle parameters are accordingly defined in a self-consistent *ab initio* fashion. Let us carry this out now in detail.

The first step is to carry out the average over P in the gamma-ray distribution. [This uses Wigner's result for $D\times D$, and also the convention $(D_{m'm}{}^{k})^* = (-)^{m'-m}D_{-m',-m}{}^{k}$.]

One subsidiary result is needed:

$$\sum_P(-P)^{\sigma(\pi)+\sigma(\pi')}C_{P-P}{}^{LL'k} = 2(-)^{L+L'}C_{1-1}{}^{LL'k} \times\begin{cases} +1 & k=\text{even}, \quad \text{parity}(L\pi)=\text{parity}(L'\pi') \\ -1 & k=\text{odd}, \quad \text{opposite parity}. \end{cases} \quad (21)$$

For parity conserved and no circular polarization measurements, k = even only.

One then finds the familiar form:

$$W_\gamma(\vartheta) \propto \sum_{\substack{kq,L\pi,L'\pi'\\ M,M',M_iM_f}}[(2L+1)(2L'+1)]^{1/2} \times(-)^{L'+q+1}C_{1\,-1}{}^{LL'k}D_{0,q}{}^{k}(\mathcal{R}^{-1}) \times[(-)^{L-M}C_{M\,-M'\,q}{}^{LL'k}\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L\pi}{}^{M\dagger}|J_iM_i\rangle \times\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L'\pi'}{}^{M'\dagger}|J_iM_i\rangle^*]. \quad (22)$$

The details are a bit more complicated for the electron distribution.

The analogous result for the average over P is the subsidiary result for the average over τ. This result is

$$\sum_{\tau}[-S(\kappa)]^{\tau+\frac{1}{2}}C_{\tau-\tau}{}^{jj'k}=2\cdot[(2l+1)(2l'+1)]^{1/2}\times(-)^{j+j'-k}C_{00}{}^{ll'k}W(ljl'j';\tfrac{1}{2}k).\quad(23)$$

[This formula is BR Eq. (86), but lacks the factor of 2 on the right in that reference.] One also needs the result of the average over μ_i.

$$\sum_{\mu_i\mu\mu'}(-)^{\frac{1}{2}+\mu'}C_{\mu_iM\mu}{}^{j_iLj}C_{\mu_iM'\mu'}{}^{j_iL'j'}C_{\mu\,-\mu'q}{}^{jj'k}=(-)^{L'+q+1}(-)^{j_i-\frac{1}{2}-k}\cdot 2|\kappa\kappa'|^{1/2}\times(-)^{L-M}C_{M\,-M'q}{}^{LL'k}W(jLj'L';j_ik).\quad(24)$$

Finally one needs the definition of reduced matrix elements:

$$\langle\kappa\mu|\mathbf{j}_{\text{el}}\cdot\mathbf{A}_{L\pi}{}^{M}(\text{out})|\kappa_i\mu_i\rangle\equiv\langle\kappa|L\pi|\kappa_i\rangle C_{\mu_iM\mu}{}^{j_iLj}.\quad(25)$$

Using these results one may reduce the $W_{\text{c.e.}}(\theta)$ as indicated in the successive steps below:

a. Original form (using only Wigner's result for $D\times D$ and reduced matrix elements):

$$W_{\text{c.e.}}(\theta)\propto\sum_{\substack{kq,LM\pi;\,L'M'\pi'\\ \kappa\kappa'M_iM_f}}\Big(\sum_{\tau}[S(\kappa)S(\kappa')]^{\tau-\frac{1}{2}}(-)^{\tau+\frac{1}{2}}C_{\tau-\tau}{}^{jj'k}\Big)\times\Big(\sum_{\mu_i\mu\mu'}(-)^{\frac{1}{2}+\mu'}C_{\mu-\mu'q}{}^{jj'k}C_{\mu_iM\mu}{}^{j_iLj}C_{\mu_iM'\mu'}{}^{j_iL'j'}\Big)\times|\kappa\kappa'|^{1/2}e^{i(\Delta_\kappa-\Delta_{\kappa'})}D_{0q}{}^{k}(R^{-1})\langle\kappa|L\pi|\kappa_i\rangle\langle\kappa'|L'\pi'|\kappa_i\rangle^{*}\times\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L\pi}{}^{M\dagger}|J_iM_i\rangle\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L'\pi'}{}^{M'\dagger}|J_iM_i\rangle^{*}.\quad(26)$$

b. Substituting for terms in parentheses:

$$W_{\text{c.e.}}(\theta)\propto\sum_{\substack{kq,M_iM_f\\ LM\pi,L'M'\pi'}}((-)^{L'+q+1}D_{0q}{}^{k}(R^{-1}))\times\{(-)^{j_i-\frac{1}{2}}\sum_{\kappa\kappa'}(-)^{j+j'}(\kappa\kappa')[(2l+1)(2l'+1)]^{1/2}C_{00}{}^{ll'k}\times\exp[i(\Delta_\kappa-\Delta_{\kappa'})]W(ljl'j';\tfrac{1}{2}k)W(jLj'L';j_ik)\times\langle\kappa|L\pi|\kappa_i\rangle\langle\kappa'|L'\pi'|\kappa_i\rangle^{*}\}[(-)^{L-M}C_{M\,-M'q}{}^{LL'k}\times\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L\pi}{}^{M\dagger}|J_iM_i\rangle\langle J_fM_f|\mathbf{j}_N\cdot\mathbf{A}_{L'\pi'}{}^{M'\dagger}|J_iM_i\rangle^{*}].\quad(27)$$

Direct comparison of the correlation $W_{\text{c.e.}}$ [Eq. (27), above] *with* W_γ [Eq. (22)] *shows that the particle parameter* b_k' (*for the K shell*) *is* [the prime denotes unnormalized (i.e., $b_{k=0}\neq 1$)]

$$b_k'=([(2L+1)(2L'+1)]^{1/2}C_{1\,-1}{}^{LL'k})^{-1}\sum_{\kappa\kappa'}(-)^{j+j'}(\kappa\kappa')\times[(2l+1)(2l'+1)]^{1/2}C_{00}{}^{ll'k}W(ljl'j';\tfrac{1}{2}k)\times W(jLj'L';\tfrac{1}{2}k)\exp(i[\Delta_\kappa-\Delta_{\kappa'}])\times\langle\kappa|L\pi|-1\rangle\langle\kappa'|L'\pi'|-1\rangle^{*}.\quad(28)$$

The principal results of this examination of the internal conversion correlation parameters are contained in Eq. (28) above. It should be noted that this result has been established in a quite direct fashion and is completely defined in terms of the 'plane-wave' bases. The critical point in this derivation—the electron basis, defined by Eqs. (13), (15), (19)—will be discussed in detail in section (IV) below.

The explicit results contained in Eq. (28) are most conveniently discussed in terms of two special cases: (a) the $(Le-L'e)$ and $(Lm-L'm)$ parameters, and (b) the $(Le-L'm)$ conversion parameters.

(*a*) *Correlation parameters for e—e and m—m multipoles.* For the $e-e$ case, one has $\pi=\pi'=e$, and thus $\kappa=L$, $-L-1$ and $\kappa'=L'$, $-L'-1$. (It follows that $l(\kappa)=L$, $l(\kappa')=L'$ in the $e-e$ case.)

For the $m-m$ case, one has $\pi=\pi'=m$, and thus $\kappa=-L$, $L+1$ with $\kappa'=-L'$, $L'+1$. This situation is treated very compactly by the identity [BR Eq. (A6)]:

$$[(2l+1)(2l'+1)]^{1/2}C_{00}{}^{ll'k}W(ljl'j';\tfrac{1}{2}k)=[(2L+1)(2L'+1)]^{1/2}C_{00}{}^{LL'k}W(LjL'j';\tfrac{1}{2}k).\quad(29)$$

Thus the coefficients are precisely the same in both the $(e-e)$ and $(m-m)$ cases, and for both we have the result:

$$b_k'=(C_{00}{}^{LL'k}/C_{1\,-1}{}^{LL'k})\sum_{\kappa\kappa'}(-)^{j+j'}(\kappa\kappa')[W(LjL'j';\tfrac{1}{2}k)]^{2}\times\exp(i[\Delta_\kappa-\Delta_{\kappa'}])\langle\kappa|L\pi|-1\rangle\langle\kappa'|L'\pi'|-1\rangle^{*}.\quad(30)$$

The sum over κ, κ' in this result takes on a rather simple general form when the W coefficients are introduced. It is convenient now to introduce a more concise notation in order to make the structure of the formulas more perspicuous. Let us define:

$$\exp(i\Delta_\kappa)\langle\kappa|L\pi|-1\rangle\equiv\{|\kappa|,L\pi\}.$$

One finds then that

$$b_k'(L\pi,L'\pi)=\frac{[L(L+1)L'(L'+\mathrm{L})]^{1/2}}{2(2L+1)(2L'+1)}[\{L,L\pi\}-\{L+1,\,L\pi\}][\{L',\,L'\pi\}-\{L'+1,\,L'\pi\}]^{*}+\frac{[L(L+1)L'(L'+1)]^{1/2}}{L(L+1)+L'(L'+1)-k(k+1)}\left(\frac{L\{L,L\pi\}+(L+1)\{L+1,\,L\pi\}}{2L+1}\right)\left(\frac{L'\{L',L'\pi\}+(L'+1)\{L'+1,\,\mathrm{L}'\pi\}}{2L'+1}\right)^{*}.\quad(31)$$

The chief usefulness of this form for the b_k is to show that at the special value (which is not quite as special as the Casimir limit): $L\{L,L\pi\}+(L+1)\{L+1,\,L\pi\}=0$—for either multipole—*the correlation parameter* b_k $(L\pi,L'\pi)$ *becomes independent of k.*

For $L=L'$ the result given in Eq. (31) assumes the simpler (normalized) form:

$$b_k(L\pi,L\pi)=1+\frac{k(k+1)}{2L(L+1)-k(k+1)}\frac{2L+1}{L|\{L,L\pi\}|^2+(L+1)|\{L+1,\,L\pi\}|^2}\left|\frac{L\{L,L\pi\}+(L+1)\{L+1,\,L\pi\}}{2L+1}\right|^2. \quad (32)$$

(*b*) *The* $(L+1e-L'm)$ *conversion parameter.* Choosing $\pi=e$ the values of kappa are $\kappa=L+1,\ -L-2$; $\pi'=m$ implies $\kappa'=-L',L'+1$. [Since the γ-ray matrix elements are real, the substitution $e\rightleftarrows m$ in Eq. (27) shows that only the real part of $b_k(L+1e;\ L'm)$ actually enters.]

The identity [BR Eq. (A7)] is tailored to handle this case:

$$[(2l+1)(2l'+1)]^{1/2}C_{00}{}^{ll'k}W(ljl'j';\tfrac{1}{2}k)W(LjL'j';\tfrac{1}{2}k)$$
$$=-\tfrac{1}{2}S(\kappa)S(\kappa')C_{1-1}{}^{L+1L'k}[(2L+3)(2L'+1)]^{1/2}$$
$$\times\left(\frac{L+2}{L+1}\right)^{\frac{1}{2}S(\kappa)}\left(\frac{L'}{L'+1}\right)^{\frac{1}{2}S(\kappa')}. \quad (33)$$

It follows that:

$$b_k'(L+1e,L'm)$$
$$=[2(2L+3)(2L'+1)]^{-1}\sum_{\kappa\kappa'}(-)^{j+j'+1}|\kappa\kappa'|$$
$$\times\left(\frac{L+2}{L+1}\right)^{\frac{1}{2}S(\kappa)}\left(\frac{L'}{L'+1}\right)^{\frac{1}{2}S(\kappa')}$$
$$\times\{|\kappa|,\,L+1e\}\{|\kappa'|,L'm\}^*. \quad (34)$$

Carrying out the sum over κ,κ' one finds that

$$b_k'(L+1e,\,L'm)=-[(L')(L'+1)(L)(L+1)]^{1/2}$$
$$\times[2(2L+3)(2L'+1)]^{-1}$$
$$\times(\{L+1,\,L+1e\}-\{L+2,\,L+1e\})$$
$$\times(\{L'+1,\,L'm\}-\{L',L'm\})^*. \quad (35)$$

It is a general result that the $b_k(L\pi,L'\pi')$ conversion parameters for $\pi=\pi'$ *are independent of* k. For the special case of $(L+1,\,e)$ mixing with (L,m) we get

$$b_k'(L+1e,\,Lm)=(-)\frac{(L+1)[L(L+2)]^{1/2}}{2(2L+1)(2L+3)}$$
$$\times(\{L+1,\,L+1e\}-\{L+2,\,L+1e\})$$
$$\times(\{L+1,\,Lm\}-\{L,Lm\})^*. \quad (36)$$

Results of the generality of Eqs. (28), (31), and (35) do not appear to have been given explicitly in the literature hitherto.

III. THE REDUCED MATRIX ELEMENTS IN TERMS OF TABULATED INTEGRALS

All relevant phases have been determined in the preceding discussion and we can now turn to the explicit evaluation, as radial Coulomb integrals, of the reduced matrix elements [defined by Eq. (25) above] of the irregular electromagnetic multipoles.

Consider first the magnetic multipoles. The irregular magnetic multipole has the operator form

$$\mathbf{A}_{Lm}{}^M(\text{out})\equiv h_L\mathfrak{Y}_{LL}{}^M$$
$$=[L(L+1)]^{-1/2}h_L(kr)\mathbf{L}(i^LY_L{}^M), \quad (37)$$

and using the standard results of angular momentum algebra, i.e.,

$$-S(\kappa)\langle\Phi_\kappa{}^\mu|(\boldsymbol{\sigma}\cdot\mathbf{L}i^LY_L{}^M)|\Phi_{-1}{}^{\mu_i}\rangle$$
$$=\langle\Phi_{-\kappa}{}^\mu|(\boldsymbol{\sigma}\cdot\mathbf{L}i^LY_L{}^M)|\Phi_1{}^{\mu_i}\rangle$$
$$=(4\pi)^{-1/2}(\kappa-1)C_{M\mu_i\mu}{}^{L\frac{1}{2}j}(\delta_\kappa{}^{-L}+\delta_\kappa{}^{L+1}), \quad (38)$$

one finds that

$$\langle\kappa|Lm|-1\rangle=[4\pi L(L+1)]^{-1/2}(\kappa-1)(-)^{L+\frac{1}{2}-j}$$
$$\times S(\kappa)(R_3'+R_4')(\delta_\kappa{}^{-L}+\delta_\kappa{}^{L+1}). \quad (39)$$

Here we have introduced the Goertzel and Rose definition of the radial integrals (Mon. P-426, 13 XI 47, Oak Ridge, Tenn.) later computed and published.[7] For the magnetic case these integrals are:

$$R_3'=\int_0^\infty r^2dr f_\kappa h_L g_{-1}{}^K, \quad (40a)$$

$$R_4'=-\int_0^\infty r^2dr g_\kappa h_L f_{-1}{}^K, \quad (40b)$$

where h_L is the spherical Hankel function of the first kind, and $f_{-1}{}^K$, $g_{-1}{}^K$ are the K-shell radial functions:

$$\begin{pmatrix}f_{-1}{}^K\\ g_{-1}{}^K\end{pmatrix}=2^{\gamma_0}(\alpha Zr)^{\gamma_0-1}$$
$$\times e^{-\alpha Zr}\left[\frac{(\alpha Z)^3}{\Gamma(2\gamma_0+1)}\right]^{1/2}\begin{pmatrix}(1-\gamma_0)^{1/2}\\(1+\gamma_0)^{1/2}\end{pmatrix}, \quad (41)$$
$$\gamma_0\equiv|(1-(\alpha Z)^2)^{1/2}|.$$

For the electric multipoles one recalls the famous 'gauge problem' (discussed by Kramer,[2] by Rose and Green,[3] and by Church and Weneser[4]) and thus transforms from the radiation gauge into the more convenient 'least singular gauge' for the point nucleus problem. This transformation is an identity transformation, and has the effect of (1) removing the term in $\mathbf{A}_{Le}{}^M$ involving h_{L+1}, (2) increasing the size of the h_{L-1} contribution

[7] M. E. Rose, G. H. Goertzel, B. I. Spinrad, J. Harr, and P. Strong, Phys. Rev. 83, 79 (1951).

so that

$$\left(\frac{L+1}{2L+1}\right)^{1/2} \rightarrow \left(\frac{2L+1}{L+1}\right)^{1/2}$$

and (3) introducing a scalar potential.

The relevant phase and normalization questions for the reduced electric multipole matrix elements can therefore be settled by considering only the vector potential term in h_{L-1}. That is,

$$\mathbf{A}_{Le}{}^{M}(\text{out}) \rightarrow h_{L-1}(kr)\left[\frac{2L+1}{L+1}\right]^{1/2} \mathfrak{Y}_{L,L-1}{}^{M} + \cdots. \quad (42)$$

To proceed further we must use the identity:

$$\mathfrak{Y}_{L,L-1}{}^{M} = i^{L-1}\{[L/(2L+1)]^{1/2}\hat{r} - [L(2L+1)]^{-1/2} i\hat{r}\times\mathbf{L}\} Y_L{}^M. \quad (43)$$

The result of these considerations is that the over-all phase and normalization may be determined from looking at that part of the electric multipole given by:

$$\langle\kappa\mu|\mathbf{j}_{\text{el}}\cdot\mathbf{A}_{Le}{}^{M}(\text{out})|-1\mu_i\rangle \rightarrow L\langle\kappa\mu|\rho_1 h_{L-1} i^{L-1}[L(L+1)]^{-1/2}\boldsymbol{\sigma}\cdot\hat{r}Y_L{}^M|-1\mu_i\rangle$$
$$= [4\pi L(L+1)]^{-1/2} L\, C_{M\mu_i\mu}{}^{L\frac{1}{2}j} \times\left[-S(\kappa)\int_0^\infty r^2 dr f_\kappa h_{L-1} g_{-1}{}^{K} - \int_0^\infty r^2 dr g_\kappa h_{L-1} f_{-1}{}^{K}\right] \times(\delta_\kappa{}^{L}+\delta_\kappa{}^{-L-1}). \quad (44)$$

[This was obtained by using the results

$$S(-\kappa)\langle\Phi_{-\kappa}{}^{\mu}|\boldsymbol{\sigma}\cdot\hat{r} i^{L-1} Y_L{}^M|-1\mu_i\rangle = \langle\Phi_\kappa{}^\mu|i^L Y_L{}^M|-1\mu_i\rangle = [4\pi]^{-1/2} C_{M\mu_i\mu}{}^{L\frac{1}{2}j}(\delta_\kappa{}^L+\delta_\kappa{}^{-L-1}).]$$

This result verifies the phase and normalization conventions and shows that the reduced matrix element for the electric multipole transitions is

$$\langle\kappa|Le|-1\rangle = [4\pi(L)(L+1)]^{-1/2}(-)^{L+\frac{1}{2}-j} \times[L(R_1+R_2+R_3+R_4)-(\kappa+1)(R_3+R_4)] \times(\delta_\kappa{}^L+\delta_\kappa{}^{-L-1}), \quad (45)$$

where the Goertzel and Rose definition of the radial integrals has been used. This definition is explicitly

$$R_1 = -\int_0^\infty r^2 dr f_\kappa h_L f_{-1}{}^K, \quad (46a)$$

$$R_2 = \int_0^\infty r^2 dr g_\kappa h_L g_{-1}{}^K, \quad (46b)$$

$$R_3 = \int_0^\infty r^2 dr f_\kappa h_{L-1} g_{-1}{}^K, \quad (46c)$$

$$R_4 = -\int_0^\infty r^2 dr g_\kappa h_{L-1} f_{-1}{}^K. \quad (46d)$$

It is useful now to compare the reduced matrix elements given in Eqs. (39) and (45), above, to the matrix elements defined in Refs. 1 and 2. In these references, the multipole potentials were not normalized to unit flux, unlike the case above. This constitutes a renormalization which affects the $b_k(L\pi,L'\pi')$ as an over-all scale change, differing, however, for each L, L'. *For the* b_k *normalized to* $b_0=1$ *there is no net effect*, but such a renormalization is well defined only for $L\pi=L'\pi'$.

The reduced matrix elements of these references are [RBA Eqs. (41) and (43)]

(a) $$Q(\kappa Lm) \equiv i[4\pi]^{-1/2}(\kappa-1)(R_3'+R_4') \times(\delta_\kappa{}^{-L}+\delta_\kappa{}^{L+1}). \quad (47)$$

Hence,

$$\langle\kappa|Lm|-1\rangle = \{i[L(L+1)]^{1/2}\}^{-1} \times(-)^{L+\frac{1}{2}-j} S(\kappa) Q(\kappa Lm). \quad (48)$$

(b) $$Q(\kappa Le) \equiv i[4\pi]^{-1/2}[L(R_1+R_2+R_3+R_4) -(\kappa+1)(R_3+R_4)](\delta_\kappa{}^L+\delta_\kappa{}^{-L-1}). \quad (49)$$

Hence,

$$\langle\kappa|Le|-1\rangle = \{i[L(L+1)]^{1/2}\}^{-1}(-)^{L+\frac{1}{2}-j} Q(\kappa Le). \quad (50)$$

IV. RESUME OF RELATIVISTIC COULOMB FUNCTIONS

The relativistic Coulomb functions to be presented here are hardly new, and have been discussed very many times since Darwin's original development in 1928. The work of Rose (Ref. 8) has furnished a standard summary since 1937. Unfortunately this used the Bethe phase convention[9] for the spherical harmonics (instead of the Condon-Shortley phase) but this is of no real consequence for the radial functions. The use of the Pauli spherical spinors ($\chi_\kappa{}^\mu$) with the standard (Rose) radial functions was initiated in RBA and has been discussed in greater detail in Ref. 10. Unfortunately, these discussions are nonetheless ambiguous—as will be clear from Sec. V. It is, therefore, still of value to resummarize the relativistic Coulomb functions in a detailed and unambiguous manner. To be more useful both the Pauli spherical spinors ($\chi_\kappa{}^\mu$) and the Pauli spherical spinors using convention T (the $\Phi_\kappa{}^\mu$) will be explicitly given.

The Dirac equation is taken in the form:

$$(\boldsymbol{\sigma}\cdot\mathbf{p}+\beta m_0 c+(E-V)/c)\psi=0. \quad (51)$$

The usual convention $\hbar=m_0=c=1$ is henceforth adopted, and the (attractive) Coulomb potential is taken to be $V=-Ze^2/r=-\alpha Z/r$. In the absence of fields, the plane wave solution for motion along the z axis may be written as $\Psi_\tau=e^{ipz}D_\tau$, where D_τ—

[8] M. E. Rose, Phys. Rev. **51**, 484 (1937).

[9] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Academic Press Inc., New York, 1957), see p. 344 ff.

[10] M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961), see p. 191 ff.

defined in Eq. (19)—are the Dirac spinors for helicity τ. The D_τ are normalized such that: $D^\dagger D=2E/(E+1)$.

The angular momentum eigenfunctions of (51) are

$$|\kappa\mu\rangle\equiv\begin{pmatrix} S(\kappa)f_\tau(pr)\Phi_{-\kappa}{}^\mu \\ g_\kappa(pr)\Phi_\kappa{}^\mu \end{pmatrix}=i^{l(\kappa)}\begin{pmatrix} -if_\kappa\chi_{-\kappa}{}^\mu \\ g_\kappa\chi_\kappa{}^\mu \end{pmatrix}, \quad (52)$$

where the latter form (without the $i^{l(\kappa)}$) corresponds to RBA Eq. (15a).

The index κ determines both the orbital angular momentum l and the total angular momentum j by the relations: (a) $j=|\kappa|-\frac{1}{2}$ (b) $l=|\kappa|+\frac{1}{2}[S(\kappa)-1]$, [where $S(\kappa)$ is a function denoting the sign of κ]. The angle functions $\Phi_\kappa{}^\mu$ are two-component spinors defined by:

$$\Phi_\kappa{}^\mu=\sum_\tau C^l{}_{\mu-\tau}{}^{1/2}{}_\tau{}^j(i^lY_l{}^{\mu-\tau})\chi_{1/2}{}^\tau, \quad (53)$$

with the 'time-reversal' phase (convention T) such that

$$T\Phi_\kappa{}^\mu=(-)^{j-\mu}\Phi_\kappa{}^{-\mu}, \quad T\equiv -i\sigma_yK_0. \quad (54)$$

$\left[\text{Note that } \chi_{1/2}{}^{1/2}\equiv\begin{pmatrix}1\\0\end{pmatrix} \text{ and } \chi_{1/2}{}^{-1/2}\equiv\begin{pmatrix}0\\1\end{pmatrix}.\right]$ From the definition it follows that $\Phi_\kappa{}^\mu$ possesses the eigenvalues $J^2\to j(j+1)$, $L^2\to l(l+1)$, $J_z\to\mu$. Moreover, one has the eigenvalue equation

$$-(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\Phi_\kappa{}^\mu=\kappa\Phi_\kappa{}^\mu, \quad (55)$$

and the relation

$$i\boldsymbol{\sigma}\cdot\hat{r}\Phi_\kappa{}^\mu=S(\kappa)\Phi_{-\kappa}{}^\mu. \quad (56)$$

The spherical spinors $\chi_\kappa{}^\mu$ are related to the $\Phi_\kappa{}^\mu$ by: $\Phi_\kappa{}^\mu=i^{l(\kappa)}\chi_\kappa{}^\mu$. Hence the $\chi_\kappa{}^\mu$ obey the same eigenvalue equations, but differ in their behavior under time reversal and under the operator $\boldsymbol{\sigma}\cdot\hat{r}$. Explicitly:

$$T\chi_\kappa{}^\mu=S(\kappa)(-)^{1/2+\mu}\chi_\kappa{}^{-\mu}, \quad (57)$$

and

$$\boldsymbol{\sigma}\cdot\hat{r}\chi_\kappa{}^\mu=-\chi_{-\kappa}{}^\mu. \quad (58)$$

For the radial functions one has, for positive energy states, the results:

$$f_\kappa=-(p/\pi)^{1/2}(E-1)^{1/2}C(\gamma,\eta)(pr)^{\gamma-1}\,\mathrm{Im}(\Lambda), \quad (59a)$$

$$g_\kappa=(p/\pi)^{1/2}(E+1)^{1/2}C(\gamma,\eta)(pr)^{\gamma-1}\,\mathrm{Re}(\Lambda), \quad (59b)$$

where

$$\Lambda\equiv(\gamma+i\eta)e^{i(\varphi-pr)}{}_1F_1(\gamma+1+i\eta,2\gamma+1,2ipr), \quad (60a)$$

$$\gamma\equiv|(\kappa^2-(\alpha Z)^2)^{1/2}| \quad (60b)$$

$$\eta\equiv\alpha ZE/p \quad \text{(positive for electrons)} \quad (60c)$$

$$C(\gamma,\eta)\equiv 2^\gamma e^{\frac{1}{2}\pi\eta}|\Gamma(\gamma+i\eta)|/\Gamma(2\gamma+1) \quad (60d)$$

$$e^{2i\varphi}\equiv e^{+\pi i}\left(\frac{\kappa-(i\alpha Z/p)}{\gamma+i\eta}\right), \quad (60e)$$

$$\text{with} \quad 0\leqslant\varphi\leqslant\pi/2 \quad \kappa>0$$

$$-\frac{\pi}{2}\leqslant\varphi\leqslant 0 \quad \kappa<0 \quad \text{(for electrons)}.$$

These definitions have the consequence[11] that the asymptotic form of the radial functions ($pr\to\infty$) is given by

$$f_\kappa\sim[p(E-1)/\pi]^{1/2}S(\kappa)j_{l(-\kappa)}(pr+\Delta_\kappa), \quad (61)$$

$$g_\kappa\sim[p(E+1)/\pi]^{1/2}j_{l(\kappa)}(pr+\Delta_\kappa), \quad (62)$$

where the relativistic Coulomb phase shift is given by

$$\Delta_\kappa\equiv\eta\ln 2pr+\varphi-\arg\Gamma(\gamma+i\eta)+\frac{\pi}{2}(l(\kappa)-S(\kappa)-\gamma). \quad (63)$$

In terms of the phase shift δ_κ defined by Rose one has [BR Eq. (94)],

$$\Delta_\kappa=\delta_\kappa+\frac{\pi}{2}(l(\kappa)-S(\kappa)). \quad (64)$$

The phase shift Δ_κ is defined relative to the plane wave result and hence $\Delta_\kappa=0$ for $Z=0$. Thus, one finds that $\delta_\kappa(Z=0)=\frac{1}{2}\pi[S(\kappa)-l(\kappa)]$, which checks with the definition of φ given above.

[Note that the asymptotic form in terms of spherical Bessel functions is *formal* (since the range of the Coulomb potential is larger than that of the centrifugal potential) but nevertheless convenient since the spherical functions are the plane-wave basis. Note, too, that it is customary (Breit[12]) to omit the logarithmic phase ($\eta\ln 2pr$) when giving the nonrelativistic Coulomb phase shift.]

The special case $Z=0$ is given by the results:

$$f_\kappa\to(-)[p(E-1)/\pi]^{1/2}j_{l(-\kappa)}(pr), \quad (65a)$$

$$g_\kappa\to S(-\kappa)[p(E+1)/\pi]^{1/2}j_{l(\kappa)}(pr). \quad (65b)$$

The analog of a plane wave, helicity τ, incident along the z axis in the presence of a charge Z at the origin, is given by

$$|p\hat{z},\tau\rangle=[\pi/(2pE)]^{1/2}\sum_\kappa(4\pi|\kappa|)^{1/2}[-S(\kappa)]^{\tau-1/2}\times\exp(i\Delta_\kappa)|\kappa\tau\rangle, \quad (66)$$

where the spherical eigenfunctions $|\kappa\tau\rangle$ are defined above. The normalization corresponds to a plane wave, that is,

$$|p\hat{z},\tau\rangle\sim D_\tau\exp(i\text{“}pz\text{”})+\text{outgoing spherical waves},$$

where "pz" corresponds to the device whereby $pr\to pr+\eta\ln 2pr$ in the asymptotic expressions.

V. COMPARISON TO PUBLISHED RESULTS

When the necessary substitutions are made to compare with previously published special case [this re-

[11] *Higher Transcendental Functions*, edited by A. Erdely (McGraw-Hill Book Company, Inc., New York, 1955), Vol. I.

[12] G. Breit and M. H. Hull, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 41.

quires explicitly the phase $\delta_\kappa(Z=0)$ given above] one finds the following results:

(a) The conversion correlation coefficients $b_k(LLe)$ [RBA Eq. (46), (46a), BR Eq. (94), (95a)] are obtained precisely. The correlation coefficients $b_k(LLm)$ given by RBA (48) and BR (96b), (97) are also obtained precisely: [RBA Eq. (48a) (defining T_m) is off by a minus sign; however, this is but a misprint in the formula only.]

(b) A similar determination of the $b_k(LL\pi)$ [Young,[13] Eq. (19)] based on the same Goertzel-Rose matrix elements, is, however, *off by a sign in the cross-term* (equivalent to $T_e\rightarrow -T_e$, $T_m\rightarrow -T_m$].

(c) Comparison of Eq. (36) to BR Eq. (100) shows that these results are in complete agreement.

(d) There is, however, *a missing minus sign in* BR *Eq. (101)*. [The original manuscript contains an explicit verification of the (correct) sign. Unfortunately, the wrong sign was transcribed into Oak Ridge National Laboratory Report ORNL 1324—which appeared between the manuscript and the galley proofs and thus set the standard.]

Consequently, Church, Schwarzschild, and Weneser are entirely correct and BR *Table IV has the incorrect sign everywhere.*

(e) The results of the Casimir approximation given in BR p. 755 has an incorrect relative phase, and as a result the high-energy limit is off by a minus sign.

In the limit of high energies the K-shell conversion correlation approaches that for a gamma ray, assuming no polarization measurements. Interestingly enough this is *still* true for longitudinal polarization measurements (on both the γ-ray and the conversion electron) in the limit of high energies. [The parallel between γ's and high-energy conversion necessarily breaks down for transverse polarization; the $b_k(\text{transverse})\rightarrow 0$ for $E\rightarrow\infty$.]

Concluding Remarks

The preceding results probably will appear quite satisfactory since one conclusion is simply to confirm the sign stated by Church, Schwarzschild, and Weneser. To rest content with such a conclusion would be quite superficial, however, since a very puzzling sign ambiguity is still concealed in this check. This is clear from the fact that the work of Young (Ref. 13)—which was a critical analysis of internal conversion correlations including polarization measurements—is in *disagreement* with the previous check, and yet using precisely the same Goertzel-Rose definitions for the radial integrals manages nonetheless to agree with tabulated low-energy numerical results!

[13] R. C. Young, Phys. Rev. **115**, 582 (1959).

The root of this difficulty stems from the fact that the definition of the phase φ [Eq. (60e) above] is ambiguous in Ref. 3 and Ref. 4, Eq. (5) p. 192, since the proper quadrant for φ is not specified and not determined by the definition given in these references. (The ambiguity appears in all texts we have examined, but we cannot claim a complete survey.)

The calculations and results of Young are *not incorrect* (despite the disagreement noted), since they are carried out in a self-consistent fashion with an explicitly defined phase convention for φ: $\varphi_{\text{Young}}=\varphi_{\text{above}}+\frac{1}{2}\pi(1+S(\kappa))$. [The phase Δ_κ given in Ref. 12, Eqs. (9), (10) seems to be misprinted in sign, however.]

The preceding work shows that angular correlation calculations are optimally defined in terms of the 'plane wave' results. From this point of view, the definitions used by Young are in fact preferable, since an annoying factor of $S(-\kappa)$ is eliminated from the radial wave functions (cf. the $Z\rightarrow 0$ limit). One has a choice of *either* phase convention for the phase φ, since the only requirement is consistency. The choice of phase used in RBA and in BR—explicitly given in Eq. (60e) above—was not made clear in these references; even though it is not the best choice, we have chosen it as the standard here to necessitate as few changes as possible in previously published results.

Finally, it should be noted that the Goertzel-Rose integrals themselves are not well defined until the phase ambiguity is explicitly removed [cf. Eq. (23), Ref. 5]. The phase choice given above coincides with the phase actually used in the numerical evaluation of these integrals.

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